Note on Determinants and Duadic Synthemes.

By J. J. Sylvester.

(Continuation. See pp. 89–96 of this Volume.)

The properties of the ω series 1, 1, 2, 8, 50, . . . (see p. 94) present some features of interest. These are the numbers of distinct terms in pure skew determinants of the order $2n$ divided by the product of the odd integers inferior to $2n$. Such numbers themselves may be termed the denumerants, and the quotients, when they are so divided, the reduced denumerants of the corresponding determinants; or for greater brevity we may provisionally call these reduced denumerants *skew numbers*. We have found, in what precedes, that

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{1-t}} = \omega_0 + \omega_1 \frac{t}{2} + \omega_2 \frac{t^2}{2.4} + \omega_3 \frac{t^3}{2.4.6} + \ldots$$

From this we may easily obtain

$$\omega_x = \frac{Fx}{2^x},$$

where $Fx = 1 + 1.x + 1.5 \frac{x(x-1)}{1.2} + 1.5.9 \frac{x(x-1)(x-2)}{1.2.3} + \ldots + \ldots$ $\ldots + 1.5.9 \ldots (4x-3)$, which shows that Fx, for all values of x, contains 2^x as a factor, and that if we take x greater than unity, 2^{x+1} will be a factor of Fx. In general, it follows from the fundamental equation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$ that if two consecutive skew numbers ω_c, ω_{c+1} have a common factor, all those of superior orders, and consequently $\frac{Fx}{2^x}$, for all values of x from c upwards, will contain such factor. It becomes then a matter of interest to assign, if possible, a general expression for the greatest common measure of ω_x, ω_{x+1}.

In the first place I say these can have no common odd factor other than unity.

Lemma. It is well known that, in the development of $(1+a)^x$, all the coefficients except the first and last will contain x when it is a prime number. More generally it may easily be shown (and the mode of proof* is too obvious to need setting out) that whatever x may be, any prime number contained in it must either divide any number r, or else the coefficient of a^r in the binomial expression above referred to. Hence we may prove that ω_x and x cannot have a common odd factor other than unity. For if possible, let $x=qp$, where p is a prime number contained in ω_x. Let the qp terms in Fx subsequent to the first term be divided into q groups, each containing p terms. Each of the terms in any one group (except the last) contains a binomial coefficient, which, by virtue of the lemma, will contain p. Moreover, the last term in the kth group will contain the factor $1.5.9\ldots(4kp-1)$.

If p is of the form $4n-3$, the nth term of the series 1, 5, 9, . . . will be p, and if it is of the form $4n-1$, the $(3n)$th term will be $3p$; and as $\frac{p+3}{4}$ and $3\,\frac{p+1}{4}$ are each not greater than p (and *à fortiori* not greater than kp) when p is greater than 1, it follows that the last coefficient, as well as all the others in any group, contains p. Hence $Fx=pP+1$, and therefore ω_x, *i. e.* $\frac{Fx}{2^x}$, cannot contain p. Hence the greatest common measure of ω_x and ω_{x+1} is a power of 2.

It will presently be shown by induction (waiting a strict proof)† that $\frac{\omega_{4x-2}}{2^x}$, $\frac{\omega_{4x-1}}{2^x}$, $\frac{\omega_{4x}}{2^x}$, $\frac{\omega_{4x+1}}{2^x}$ are all of them integers, and the first, third and fourth, odd integers; from this it will easily be seen that the greatest common measure of ω_x, ω_{x+1} is $2^{\theta\left(\frac{2x+1}{8}\right)}$, where, in general, $\theta(\mu)$ means the integer

*Some of the prolixity of the more obvious mode of proof of this lemma may be avoided by the substitution of the following method:

$$\text{Call } (1+t)^n = 1 + A_1t + A_2t^2 + A_2 + A_3t^3 + \ldots, \text{ so that}$$
$$n(1+t)^{n-1} = A_1 + 2A_2t + 3A_3t^2 + \ldots$$
$$= B_0 + B_1t + B_2t^2 + \ldots\ldots = \phi t.$$

Suppose $n=qp$: then designating the qth roots of unity by $\rho_1, \rho_2 \ldots \rho_q$, we have

$$\frac{1}{q}\,\Sigma\rho^{q-k}\,\phi(\rho t) = B_kt^k + B_{k+q}t^{k+q} + B_{k+2q}t^{k+2q} + \ldots + B_{k+(p-1)q}t^{k+(p-1)q},$$

and the left hand side of the equation is obviously a multiple of p. Hence, putting t successively equal to 0, 1, 2, 3, . . . $(p-1)$, we obtain, by a well-known theorem of determinants,

$$\Delta B_{k+\lambda q} \equiv 0 \;[\bmod p],$$

where Δ, being the product of the differences of 0, 1, 2, . . . $(p-1)$, cannot contain p. Hence $B_{k+\lambda q}\equiv 0$ [mod p], and consequently giving k all values from 0 to $(q-1)$, and λ all values from 0 to $(p-1)$, we see that all the B's, from B_0 to B_{pq-1}, must contain p as a factor as was to be proved.

† Since the above was set up in print, I have found an easy proof, for which see *Postscript*.

nearest to μ. Let us call the above fractions q_{4x-2}, q_{4x-1}, q_{4x}, q_{4x+1}, to which we may give the name of simplified skew numbers. In the subjoined table I have calculated the values of the residues of these numbers by a regular algorithm in respect to *moduli* beginning with 2^{23} and regularly decreasing according to the descending powers of 2. R stands for the words *residue of.*

Modulus.	x	Rq_{4x-2}	Rq_{4x-1}	Rq_{4x}	Rq_{4x+1}
8,388,608	0			1	1
4,194,304	1	1	4	25	209
2,097,152	2	1,087	13,504	194,951	1,088,983
1,048,576	3	929,451	442,068	992,179	576,715
524,288	4	287,913	118,168	393,089	71,201
262,144	5	201,913	14,228	126,417	179,945
131,072	6	51,071	56,656	46,407	127,767
65,536	7	56,531	24,452	15,131	46,739
32,768	8	12,521	29,928	22,753	29,729
16,384	9	14,289	5,412	15,209	14,305
8,192	10	1,119	2,784	4,063	4,751
4,096	11	3,283	3,156	2,331	3,059
2,048	12	1,721	1,632	425	1,801
1,024	13	913	84	1,001	385
512	14	215	240	479	239
256	15	91	132	99	219
128	16	81	8	9	9
64	17	41	36	1	57
32	18	23	0	31	15
16	19	3	4	11	3
8	20	1	0	1	1
4	21	1	0	1	1
2	22	1	0	1	1

From this table it appears that q_{8i-5} is 4 times an odd number, and that q_{8i-1} is 8 times a number which may be odd or even; thus we know the exact

number of times that 2 will divide out all the skew numbers other than those whose orders are of the form $8i-1$, and an inferior limit to that number for that case.

It will further be noticed that, when x is of the form $4i$, or $4i+1$, the simplified skew numbers q_{4x-2}, q_{4x}, q_{4x+1} are all of the form $8\lambda+1$, that when x is of the form $4i+2$ the above named simplified skew numbers are of the form $8\lambda+7$, and when x is of the form $4i+3$, they are of the form $8\lambda+3$.

Before quitting this subject, I think it desirable briefly to refer to other series of integers closely connected with those which I have called *skew numbers.* To this end we may write, in general,

$$e^{\frac{t}{4}}(1-t)^{\frac{4\mu-1}{4}}=1+\omega_{1,\mu}\frac{t}{2}+\omega_{2,\mu}\frac{t^2}{2.4}+\omega_{3,\mu}\frac{t^3}{2.4.6}+\ldots,$$

μ being any positive or negative integer, so that $\omega_{x,0}$ is the same as I have called hitherto ω_x. It may then easily be shown that $\omega_{x,\mu+1}=\dfrac{2\omega_{x+1,\mu}-\omega_{x,\mu}}{4\mu+1}$, that $\omega_{x,\mu-1}=\omega_{x,\mu}-2x\omega_{x-1,\mu}$, and that the equation in differences for $\omega_{x.\mu}$, for μ constant, becomes

$$\omega_{x,\mu}=(2x+2\mu-1)\,\omega_{x-1,\mu}-(x-1)\,\omega_{x-2,\mu},$$

with the initial conditions $\omega_{0,\mu}=1$, $\omega_{1,\mu}=2\mu+1$. Also, it is clear from the definition, that the explicit value of $\omega_{x,\mu}$ in a series becomes

$$\frac{1}{2^x}\left\{1+(4\mu+1)\,x+(4\mu+1)(4\mu+5)\,x\frac{x-1}{2}\right.$$
$$\left.+(4\mu+1)(4\mu+5)(4\mu+9)\,x\frac{x-1}{2}.\frac{x-2}{3}+\ldots\right\},$$

which is easily seen to verify the equation

$$2\omega_{x,\mu}-\omega_{x-1,\mu}=(4\mu+1)\,\omega_{x-1,\mu+1}.*$$

We might call the $\omega_{x,\mu}$ series skew numbers of the μth degree, and, as for the case of $\mu=0$, so it may be shown in general that two consecutive skew numbers of the same degree can have no common odd factor. Also, it remains true that the greatest common factor of any two consecutive skew numbers of the same degree and the orders x, $x+1$, is $2^{\theta\left(\frac{2x+1}{8}\right)}$; $\omega_{4x-2,\mu}$, $\omega_{4x-1,\mu}$, $\omega_{4x,\mu}$, $\omega_{4x+1,\mu}$ being all divisible by 2^x, and the resulting quotients being, the first, third

*And of course, in general, the equation

$$\lambda u_{x,y}-u_{r-1,y}+\phi y u_{x-1,y+\delta}=0,$$

with the condition that $u_{0,y}$ is constant, has for its integral

$$u_{x,y}=\frac{c}{\lambda^x}\left\{1-\phi yx+\phi y\phi(y+\delta)\,x\frac{x-1}{2}-\phi y\phi(y+\delta)\,\phi(y+2\delta)\,x\frac{x-1}{2}\frac{x-2}{3}+\ldots\right\}.$$

and fourth of them, always odd integers, and the second divisible by 4 or some higher power of 2 when μ is even, but only by the first power of 2 when μ is odd. But it would carry me too far away from the original object of this note, and from other investigations of more pressing moment to myself, to pursue further the theory of general skew numbers, which, however, seems to me to be well worthy of the study of arithmeticians.

I will only stop to point out that the rule for the greatest common measure of ω_x and ω_{x+1}, serves to prove the rule for the general case of $\omega_{x,\mu}$ and $\omega_{x+1,\mu}$. Thus suppose μ to be positive. Then since $\omega_{k,1} = 2\omega_{k+1} - \omega_k$, and $\omega_{4k-2} = 2^k(2\lambda+1)$, $\omega_{4k-1} = 2^{k+1}\tau$, $\omega_{4k} = 2^k(2\nu+1)$, $\omega_{4k+1} = 2^k(2\pi+1)$, $\omega_{4k+2} = 2^{k+1}(2\rho+1)$; it follows that
$\omega_{4k-2,1} = 2^k(2\lambda'+1)$, $\omega_{4k-1,1} = 2^{k+1}\tau'$, $\omega_{4k,1} = 2^k(2\nu'+1)$, and $\omega_{4k+1,1} = 2^k(2\pi'+1)$.

It is obvious further that, τ being even, τ' is odd. So again from these results we may, in like manner, deduce $\omega_{4k-2,2} = 2^k(2\lambda''+1)$, $\omega_{4k-1,2} = 2^{k+1}\tau''$, $\omega_{4k,2} = 2^k(2\nu''+1)$, $\omega_{4k+1,2} = 2^k(2\pi''+1)$, subject also to the remark that, τ' being odd, τ'' is even, and so on continually, τ being alternately even and odd. Again if μ is negative, we may, in like manner, by means of the formula $\omega_{k,\mu-1} = \omega_{k,\mu} - 2k\omega_{k-1,\mu}$, pass successively from the case of ω_k to that of $\omega_{k,-1}$: $\omega_{k,-2}$: . . . $\omega_{k,-\mu}$, and establish precisely the same conclusion in regard to powers of 2 as for the case of μ positive, and it will be remembered that I have already shown how to establish that $\omega_{k,\mu}$ and $\omega_{k+1,\mu}$ have no common *odd* factor.

In the first note on this subject (Vol. II, No. 1, of the *Journal*) I showed how a general determinant could be completely represented by means of systems of cycles and that accordingly the terms in the total development would split up into families, as many in number as there are indefinite partitions of the index of the order of the determinant—the particular mode of aggregation depending upon the term chosen to represent the product of the elements in the principal diagonal, so that for the order n there would be $1.2.3\ldots n$ distinct modes of distribution into families. This gives rise to a theory of transformation of cycles, corresponding to a transposition of the rows or columns of the matrix. Thus *ex. gr.* suppose the *umbrae* to be 1, 2, 3, . . . n: r, s signifying the element in the rth row and sth column. Then if we interchange the mth and nth columns, this will have the effect of changing pm into pn and pn into pm.

Suppose now that a term of the developed determinant is expressed by a system of cycles such that m and n lie in two distinct cycles, say Xm and nY, where X, Y are each of them single elements, or aggregates of single elements; then the effect of the interchange will be to bring these cycles into the single cycle $XnYm$. If Xm, nY were both odd ordered or both even ordered cycles, their sum will be even ordered, and the number of *even* cycles will be increased or diminished by unity; so if one was of odd and the other of even order, their sum will be of odd order, and the number of even cycles will be diminished by unity. In either case, therefore, the sign, which depends on the *parity* of the number of even cycles, is reversed.

Again, suppose m and n to lie in the same cycle $mXnY$. Then the effect of the interchange will be to break this up into two cycles mX, nY, and for the same reason as above the sign will be reversed. Thus the sign of every term in the development will, we see, be reversed, as we know *à priori* ought to be the case.

I shall conclude with applying the formula $\omega_x = \frac{Fx}{2^x}$ to determining the *asymptotic* mean value of the coefficients in a skew determinant of the order $2x$, *i. e.* the function of x to which the mean value of the coefficients converges when x is taken indefinitely great. We know that all the coefficients, both in this case and in that of a symmetrical determinant, are different powers of 2; to find the mean of the indices of these powers would be seemingly an investigation of considerable difficulty, but there will be little or none in finding the ultimate expression for the mean of the coefficients themselves, or, which is the same thing, the first term in the function which expresses this mean in terms of descending powers of x. We shall find that, for symmetrical determinants, this is a certain multiple of the square root and, for skew determinants, of the fourth root of x, as I proceed to show.

From the equation

$$2^x\omega_x = 1 + x + 5x\frac{x-1}{2} + \ldots + (1.5\ldots(4x-3)),$$

we have, when $x = \infty$,

$$2^x\omega_x = 1.5.9\ldots\overline{4x-3}\left\{1 + \frac{x}{4x-3} + \frac{1}{2}\,\frac{x(x-1)}{(4x-3)(4x-7)} + \ldots\right\}$$

$$= e^{\frac{1}{4}}.1.5.9\ldots\overline{4x-3}.$$

The number of terms in the Pfaffian (the square root of the determinant taken with suitable algebraical sign) being $1.3.5\ldots\overline{2x-1}$ and—as follows from

what was shown in the first note—cancelling being out of the question, the sum of the coefficients all taken positively in the determinant itself will be $(1.3.5\ldots\overline{2x-1})^2$. Hence the mean value required is $(1.3.5\ldots\overline{2x-1})^2$ divided by $1.3.5\ldots\overline{2x-1}\;\omega_x$, to express which quotient in exact terms we may make use of the formula

$$\frac{a(a+\delta)(a+2\delta)\ldots(a+x\delta)}{b(b+\delta)(b+2\delta)\ldots(b+x\delta)}=\frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}}x^{\frac{a-b}{\delta}}.$$

For the mean value is

$$\frac{1}{e^{\frac{1}{4}}}\cdot\frac{1.3.5\ldots(2x-1)}{2.4.6\ldots(2x)}\cdot\frac{4.8.12\ldots(4x)}{1.5.9\ldots(4x-3)}=\frac{1}{e^{\frac{1}{4}}}\cdot\frac{1}{\Gamma\frac{1}{2}}x^{-\frac{1}{2}}.\Gamma\frac{1}{4}x^{\frac{3}{4}}=\frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{\pi}}x^{\frac{1}{4}}.$$

If we write this under the form $Qx^{\frac{1}{4}}$, we have

$$Q=\frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\Gamma\frac{1}{2}},$$

$$\begin{aligned}\log Q&=\log\Gamma\frac{5}{4}+\log 2-\log\Gamma\frac{3}{2}-\frac{1}{4}\log e\\&=9.9573211+.3010300-9.9475449-.1085736\\&=.2022326,\\ \text{or } Q&=1.59306.\end{aligned}$$

This result as may easily be seen remains unaffected when, instead of a pure skew determinant, one is taken in which the diagonal terms retain general values. The effect of this change will be to increase the numerator and denominator of the fraction which expresses the mean value, in the proportion of $\frac{e^2+1}{2e}$ to 1.

Finally, as regards the ultimate mean value of the coefficients of symmetrical determinants. This, for one of the order x, by virtue of Professor Cayley's formula previously given, will be the reciprocal of the coefficient of t^x in $\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}$. It may readily be shown in general that, ϕt being any series of integer powers of t, the coefficient of t^x (when x becomes infinite) in $\frac{e^{\phi t}}{\sqrt{1-t}}$

is in a ratio of equality to the coefficient of t^x in $\dfrac{e^{(\phi 1)t}}{\sqrt{1-t}}$, so that in the present case this coefficient is the same as the coefficient of t^x in $\dfrac{e^{\frac{3}{4}t}}{\sqrt{1-t}}$, *i. e.* in

$$\left(1+\frac{1}{2}t+\frac{1.3}{2.4}t^2+\ldots+\frac{1.3.5\ldots(2x-1)}{2.4.6\ldots 2x}t^x+\ldots\right)$$
$$\times\left(1+\frac{3}{4}t+\left(\frac{3}{4}\right)^2\frac{t^2}{2}+\ldots+\left(\frac{3}{4}\right)^x\frac{t^x}{1.2\ldots x}\right),$$

which is obviously, when x is infinite, equal to $\dfrac{1.3.5\ldots(2x-1)}{2.4.6\ldots 2x}e^{\frac{3}{4}}$. Hence the ultimate mean value of the coefficients is $\dfrac{1}{e^{\frac{3}{4}}}\dfrac{2.4.6\ldots 2x}{1.3.5\ldots(2x-1)}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$.

For a symmetrical determinant in which all the diagonal terms are wanting, the numerator of the fraction giving the mean value becomes $e^{-1}(1.2.3\ldots x)$ and the denominator is $(1.2.3\ldots x)$ into the coefficient of t^x in $\dfrac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{1-t}}$, which is the same as in $\dfrac{e^{-\frac{1}{4}t}}{\sqrt{1-t}}$. The result then is $\dfrac{\pi^{\frac{1}{2}}e^{\frac{1}{4}}}{e}\sqrt{x}$, or $\dfrac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$ as before. It may perhaps be just worth while to notice that the *skew numbers* (the ω's of the text) may be put under the form of a determinant, the nature of which is sufficiently indicated by the annexed diagram.

1	1	0	0	0	0	0
1	3	2	0	0	0	0
0	1	5	3	0	0	0
0	0	1	7	4	0	0
0	0	0	1	9	5	0
0	0	0	0	1	11	6
0	0	0	0	0	1	13

The successive principal minors in this matrix represent the successive skew numbers of all orders from 1 to 6 inclusive.

Postscript.

Since $\omega_{x+1} = (2x+1)\,\omega_x - x\omega_{x-1}$, we have
$\omega_{x+2} = (4x^2 + 7x + 2)\,\omega_x - (2x^2 + 3x)\,\omega_{x-1}$,
$\omega_{x+3} = (8x^3 + 32x^2 + 34x + 8)\,\omega_x - (4x^3 + 15x^2 + 13x)\,\omega_{x-1}$,
$\omega_{x+4} = (16x^4 + 116x^3 + 273x^2 + 231x + 50)\,\omega_x - (8x^3 + 56x^3 + 122x^2 + 82x)\,\omega_{x-1}$.
Suppose now that, for a given value of i, $q_{4i-2} = \frac{\omega_{4i-2}}{2^i} = 2\lambda + 1$, $q_{4i-1} = \frac{\omega_{4i-1}}{2^i} = 4\mu$, $q_{4i} = \frac{\omega_{4i}}{2^i} = 2\nu + 1$ and $q_{4i+1} = \frac{\omega_{4i+1}}{2^i} = 2\rho + 1$. Call $\omega_{x+4} = E_x\omega_x - F_x\omega_{x-1}$. Then when $x \equiv \pm 2$, $F_x \equiv 4$ [mod 8], and therefore, assuming that $q_{4i-3} = \frac{\omega^{4i-3}}{2^{i-1}}$ is odd, $\frac{F_{4i-2}\,\omega_{4i-3}}{2^{i+1}}$ is odd. Also, $E_{4i-2} \equiv 462 + 50 \equiv 0$ [mod 4], and consequently $\frac{E_{4i-2}\omega_{4i-2}}{2^{i+1}}$ is even; hence $q_{4i+2} = \frac{\omega_{4i+2}}{2^{i+1}}$ is integer and odd. Again when $x = 4i - 1$, $E_x \equiv 1 - 3 + 50 \equiv 0$ [mod 4], and $F_x \equiv 122 - 82 \equiv 0$ [mod 8]; hence $q_{4i+3} = \frac{\omega_{4i+3}}{2^{i+1}}$ is an integer divisible by 4. Again, when $x = 4i$, $E_{4i} \equiv 2$ and $F_{4i} \equiv 0$ [mod 4]; hence $q_{4i+4} = \frac{\omega_{4i+4}}{2^{i+1}}$ is integer and odd; and when $x = 4i + 1$, $E_{4i+1} \equiv 2$ and $F_{4i+1} \equiv 0$ [mod 4]; hence $q_{4i+5} = \frac{\omega_{4i+5}}{2^{i+1}}$ is integer and odd.

Thus it has been shown that if it be true up to $\lambda = i$ that $\frac{\omega_{4\lambda-2}}{2^\lambda}$, $\frac{\omega_{4\lambda-1}}{2^{\lambda+2}}$, $\frac{\omega_{4\lambda}}{2^\lambda}$, $\frac{\omega_{\lambda 4+1}}{2^\lambda}$ are all integer, and the first, third and fourth odd integers, the same proposition can be affirmed for all superior values of i, and being true for ω_0, ω_1, ω_2, ω_3, the quotients corresponding to which are 1, 1, 1, 1, the theorem is true universally. It is inconceivable that it could have occurred to any human being to lay down so singular a train of induction as the one above employed, unless previously prompted to do so by an *à priori* perception of the law to be established, acquired through a preliminary study and direct inspection of the earlier terms in the series of numbers to which it applies. Here then we have a salient example (if any were needed) of the importance of the part played by the *faculty of observation* in the discovery and establishment of pure mathematical laws.